On asymptotic properties of some complex Lorenz-like systems

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Abstract

The classical Lorenz lowest order system of three nonlinear ordinary differential equations, capable of producing chaotic solutions, has been generalized by various authors in two main directions: (i) for number of equations larger than three (Curry1978) and (ii) for the case of complex variables and parameters. Problems of laser physics and geophysical fluid dynamics (baroclinic instability, geodynamic theory, etc. - see the references) can be related to this second aspect of generalization. In this paper we study the asymptotic properties of some complex Lorenz systems, keeping in the mind the physical basis of the model mathematical equations.

1 Introduction

The behaviour of the classical Lorenz system of equations (Lorenz 1963)

$$\begin{aligned} \frac{dX}{dt} &= -\sigma X + \sigma Y, \\ \frac{dY}{dt} &= rX - Y - XZ, \\ \frac{dZ}{dt} &= -bZ + XY, \end{aligned} \tag{1}$$

where $\sigma > 0, r > 0, b > 0$, is well known (Shimizu and Morioka 1978, Yorke and Yorke 1979, Franceschini 1980, Sparrow 1982, McGuinnes 1983, Schmutz and Rueff 1984). Here $X(t), Y(t), Z(t)$ are real functions of the time t and (σ, r, b) are real parameters. The goal of this study is to ascertain to what extent some fundamental properties of the system (1) are presented in its complex analogs. For comparison we choose the following properties of the system of equations (1):

1. The fixed points of (1)

$$(\overline{X}, \overline{Y}, \overline{Z}) = (0, 0, 0), \quad r > 0, \tag{2}$$

$$(\overline{X} = \overline{Y} = \pm\sqrt{b\overline{Z}}, \overline{Z} = r - 1), \quad r > 1. \tag{3}$$

2. The conditions for stability of the second fixed fixed point in (3) are

$$1 < r < r_H = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}, b + 1 < \sigma < \infty. \tag{4}$$

   At $r > r_h$ a chaos occurs.

3. If $\sigma \to \infty$ then $r_H \to \infty$ and the above fixed point remains stable for arbitrary $r > 1$. In this case

$$\begin{aligned} Y(t) &= X(t), \\ X^2 &= \frac{dZ}{dt} + bZ, \\ \frac{d^2Z}{dt^2} + (b - 2f + 2Z)\frac{dZ}{dt} - 2bZ(f - Z) &= 0, \end{aligned} \tag{5}$$

   where $f = r - 1$, so that chaos is impossible.

4. If $b = 2\sigma$ and $t \to \infty$, then

$$\begin{aligned} Y(t) &= X\frac{1}{\sigma}\frac{dX}{dt}, \\ Z(t) &= \frac{1}{2\sigma}X^2, \\ \frac{d^2X}{dt^2} + (\sigma + 1)\frac{dX}{dt} + \sigma(1 - r)X + \frac{1}{2}X^3 &= 0, \end{aligned} \tag{6}$$

   so that again chaos is impossible. The single second order autonomous equations in (5) and (6) are of known types (Lienard and Duffing respectively - Perko 1996). Their derivation can be seen below.

The Lorenz system (1) is a very simplified model of thermal convection in fluids (the Rayleigh-Benard problem, see Lorenz 1963 or Tritton 1988). Later on, problems of laser physics (see Ning and Haken, 1990 and the references therein) and of so called baroclinic instability of the geophysical flows (in the atmosphere or in the ocean) unexpectedly resulted in a system of the same

structure like (1) but in the complex domain (Fowler et al. 1983, Rauh et al. 1996)

$$
\begin{aligned}
\frac{dX}{dt} &= -\sigma X + \sigma Y, \\
\frac{dY}{dt} &= \hat{r} X - aY - XZ, \\
\frac{dZ}{dt} &= -bZ + \frac{1}{2}(X^* Y + X Y^*).
\end{aligned} \tag{7}
$$

where $i = \sqrt{-1}$, $\hat{r} = r_1 + ir_2, a = 1 - ie, (X, Y) = (X_1, Y_1) + i(X_2, Y_2)$, (σ, b, e, Z) are real quantities and * denotes complex conjugate. Another similar system originated from geophysics will be introduced in section 4.

In real variables the system (7) is equivalent to the following fifth order system of ordinary differential equations

$$
\begin{aligned}
\frac{dX_1}{dt} &= -\sigma X_1 + \sigma Y_1, \\
\frac{dX_2}{dt} &= -\sigma X_2 + \sigma Y_2, \\
\frac{dY_1}{dt} &= r_1 X_1 - Y_1 - r_2 X_2 - eY_2 - X_1 Z, \\
\frac{dY_2}{dt} &= r_1 X_2 - Y_2 - r_2 X_1 - eY_1 - X_2 Z, \\
\frac{dZ}{dt} &= -Z + X_1 Y_1 + X_2 Y_2.
\end{aligned} \tag{8}
$$

In what follows we show that in some particular cases the above system simplifies and can be integrated even analytically.

2 Particular cases of the system (7)

2.1 Case $\sigma \to \infty$

Let $\sigma \to \infty$, i.e., $\epsilon = 1/\sigma \to 0$. Then the first equation of (7) implies

$$
Y(t) = X(t), \tag{9}
$$

so that (7) reduces to

$$
\frac{dX}{dt} = (\hat{r} - a - Z)X, \tag{10}
$$

$$
\frac{dZ}{dt} = -bZ + \mid X \mid^2, \tag{11}
$$

where $\mid X \mid^2 = XX^* = X_1^2 + X_2^2 = A^2 > 0$. In real variables the above equations are equivalent to a three-dimensional system of nonlinear differential equations

$$\frac{dX_1}{dt} = (f_1 - Z)X_1 - \nu X_2,$$
$$\frac{dX_2}{dt} = (f_1 - Z)X_2 + \nu X_1,$$
$$\frac{dZ}{dt} = -bZ + X_1^2 + X_2^2, \tag{12}$$

where $f_1 = r_1 - 1$, $\nu = e + r_2$. From the first two equations of (12) we obtain easily

$$\frac{d}{dt}A^2 = 2(f_1 - Z)A^2, \tag{13}$$

Eliminating A^2 from (13) and (11) we obtain a single equation of known (Lienard) type (Perko 1996)

$$\frac{d^2Z}{dt^2} + (b - 2f_1 + 2Z)\frac{dZ}{dt} - 2bZ(f_1 - Z) = 0, \tag{14}$$

In addition by eliminating either X_1 or X_2 from the first two equations of (12) we obtain the second order equation of kind forced harmonic oscillator

$$\frac{d^2F}{dt^2} - 2(f_1 - Z)\frac{dF}{dt} + \left[\nu^2 + (f_1 - Z)^2 + \frac{dZ}{dt}\right] F = 0, \tag{15}$$

where F is equal to X_1 or to X_2. The equations (14) and (15) form a decoupled system. Having determined (analytically or numerically) $Z(t)$ from (14) then we can obtain A by $A^2 = dZ/dt + bZ$ and $X_{1,2}$ can be determined by (15). For instance

$$\overline{Z} = f_1 = r_1 - 1, \overline{A}^2 = bf_1, \quad (r_1 > 1), \tag{16}$$

are stationary ($dZ/dt = 0$) solutions (fixed points) of (13) and (11) for which (15) reduces to the simplest (linear) harmonic oscillator $d^2F/dt^2 + \nu^2 F = 0$. Hence $\overline{A} = \sqrt{bf_1}$ and

$$\overline{X_1}(t) = \overline{A}\cos(\nu t), \quad \overline{X_2}(t) = \overline{A}\sin(\nu t) \tag{17}$$

Thus (17) is a simple limit cycle solution, which is a result of a Hopf bifurcation after loss of stability at $r_1 = 1$ of the trivial solution $\overline{Z} = \mid \overline{X} \mid = 0$ of (11) and (13). Standard stability analysis of the above fixed point shows

that it is stable for entire range $r_1 > 1$ so that (17) is a stable cycle for all $r_1 > 1$.

In the general case the Lienard equation (14) can not be solved analytically. For its numerical integration a more appropriate form is

$$\frac{dZ}{dt} = U - (b - 2f_1)Z - Z^2,$$
$$\frac{dU}{dt} = 2bZ(f_1 - Z). \tag{18}$$

If $b = 2f_1$ both equations (14) and (18) simplify.

It is of interest to study the behavior of $Z(t)$ and $A^2(t)$ in (11) and (13) far from the initial instant, i.e., in the posttransient period. The general solution of (11) with respect to $Z(t)$ can be written as

$$Z(t) = Z_0 \exp(-bt) + \int_0^t d\tau \exp(-b\tau) \mid X(t-\tau). \mid^2 \tag{19}$$

Hence at $t >> \tau_b = 1/b$ (theoretically $t \to \infty$)

$$Z(t \to \infty) = Z_\infty(t) = \int_0^\infty d\tau \exp(-b\tau) \mid X(t-\tau) \mid^2 . \tag{20}$$

The substitution of (13) for $Z(t)$ yields the integrodifferential equation

$$\frac{d}{dt} \mid X^2 \mid = 2\left[f_1 - \int_0^\infty d\tau \exp(-b\tau) \mid X(t-\tau) \mid^2\right] \mid X \mid^2 . \tag{21}$$

valid at $t \to \infty$. The integral term can be interpreted as memory of the process for the past. If in addition $b >> 1$ (in the limit $b \to \infty$) , i.e., in the case of short memory, the expression (20) is reduced to

$$Z_\infty(t) = \frac{1}{b} \mid X(t) \mid^2 = \frac{1}{b} A^2(t), \tag{22}$$

so that from (21)

$$\frac{d}{dt} A^2 = 2f_1 A^2 - \frac{2}{b} A^4, \quad t \to \infty. \tag{23}$$

However (23) is exactly the Landau equation in turbulence theory (Landau and Lifshitz, 1986). Its integration is an easy task.

$$A^2(t) = A_0^2 b f_1 \left[A_0^2 + (bf_1 - A_0^2)\exp(-2f_1(t - t_0))\right]^{-1} \tag{24}$$

where $A_0^2 = [A(t_0)]^2$. Therefore, independently of A_0, $A^2(\infty) = bf_1 = \overline{A}^2$, which means that the fixed points (16) and the corresponding cycle (17) are globally stable.

2.2 Case of finite σ ($\sigma < \infty$)

Let us now assume finite σ ($\sigma < \infty$). Then from the X, Z equations (12) we derive a new one

$$\frac{dZ}{dt} + bZ = \mid X \mid^2 + \frac{1}{2\sigma}\frac{d}{dt} \mid X \mid^2, \tag{25}$$

which can be rewritten as

$$\frac{dM}{dt} + bM = \left(\frac{b}{2\sigma} - 1\right) \mid X \mid^2, \tag{26}$$

or alternatively

$$\frac{dM}{dt} + 2\sigma M = (b - 2\sigma)Z(t), \tag{27}$$

where

$$M(t) = \frac{1}{2\sigma} \mid X \mid^2 - Z. \tag{28}$$

Hence in view of (11) and (20)

$$M_\infty(t) = M(t \to \infty) = \left(\frac{b}{2\sigma} - 1\right) \int_0^\infty d\tau \exp(-b\tau) \mid X(t-\tau) \mid^2, \tag{29}$$

and

$$Z_\infty(t) = \frac{1}{2\sigma} \mid X(t) \mid^2 - M_\infty(t), \tag{30}$$

so that $M_\infty = 0$ at $b = 2\sigma$. Then the first two equations (12) combined with (30) lead to a single equation

$$\frac{d^2X}{dt^2} + (\sigma + a)\frac{dX}{dt} + \sigma(a - \hat{r})X + \frac{1}{2} \mid X \mid^2 X = 0, \tag{31}$$

valid at $t \to \infty$ and $b = 2\sigma$. Obviously, this is a generalized autonomous Duffing type equation in the complex domain. In real variables it is equivalent to two coupled equations for $X_1(t)$ and $X_2(t)$ of the same kind

$$\begin{aligned} \frac{d^2X_1}{dt^2} + (\sigma+1)\frac{dX_1}{dt} + e\frac{dX_2}{dt} - \sigma f_1 X_1 + \sigma\nu X_2 + \frac{1}{2} \mid X \mid^2 X_1 = 0, \\ \frac{d^2X_2}{dt^2} + (\sigma+1)\frac{dX_2}{dt} + e\frac{dX_1}{dt} - \sigma f_1 X_2 + \sigma\nu X_1 + \frac{1}{2} \mid X \mid^2 X_2 = 0, \end{aligned} \tag{32}$$

where $\mid X \mid^2 = X_1^2 + X_2^2$, $f_1 = r_1 - 1$, $\nu = e + r_2$ and $X = X_1 + iX_2$.

Since $a - \hat{r} = -f_1 - i\nu$, equation (31) has only one stationary solution (fixed point): $\overline{X} = 0$. No other such solutions occur unless $\nu = 0$, known as "laser case" (Ning and Haken 1990). Then

$$\mid \overline{X} \mid^2 = 2\sigma(r_1 - 1), \quad r_1 > 1, \tag{33}$$

and stationary oscillations in the lasers can exist. However, such constraint ($\nu = 0$) is not generally required for the baroclinic instability. In the latter case $\nu \neq 0$.

$$X(t) = A\exp(i\omega t), \omega = \frac{\sigma\nu}{\sigma+1}, A^2 = 2\sigma\left[f_1 - \nu\frac{e-\sigma r_2}{(\sigma+1)^2}\right], \tag{34}$$

is an exact limit cycle solution of the nonlinear equation (31) respectively (32), provided the term in the bracket is positive, i.e., $r_1 > r_{1c} = 1 + \nu(e - \sigma r_2)/(\sigma+1)^2$.

If $b \neq 2\sigma$, the memory term $\sigma M_\infty(t)$ will stand in (31) instead of zero. This changes the expression for the amplitude A^2 only: 2σ is replaced by b ($b < 2\sigma$ or $b > 2\sigma$), thus decreasing (increasing) A^2 compared to (34).

For the case of finite σ but $b \to 0$ we can use the the alternative equation (27)

$$\frac{dM}{dt} + 2\sigma M = -2\sigma Z(t).$$

Hence similarly to (29) and (30)

$$M_\infty(t) = -2\sigma\int_0^\infty d\tau\, \exp(-2\sigma\tau)Z_\infty(t-\tau),$$

and

$$\frac{1}{2\sigma} \mid X_\infty(t) \mid^2 = Z_\infty(t) + M_\infty(t).$$

Contrary to the previous case (20) the memory effect now is carried out by the phase variable $Z(t)$.

Finally, we note that equation (14) is identical to the last equation (5) and equation (31) degenerates into equation (6) in the real case ($e = r_2 = 0, X_2 = 0, X_1 = X, r_1 = r$).

3 The infinite $Z-$components version of (7)

The paper by Booty et al. (1982) contains results for the system

$$\begin{aligned}
\frac{dX}{dt} &= -\sigma X + \sigma Y, \\
\frac{dY}{dt} &= (\hat{r} - \sum_{n=1}^{\infty} Z_n)X - aY, \\
\frac{dZ_n}{dt} &= -b_n Z_n + \frac{1}{2}c_n(X^*Y + XY^*),
\end{aligned} \tag{35}$$

where $b_n, c_n > 0$, $n = 1, 2 \ldots, \infty$, $(\sigma, \hat{r}, a)$ are the same parameters as in (7). However, unlike (7), the physical basis of the above system is the baroclinic instability in the geophysical fluids only. Here we are interested in (35) from mathematical point of view.

First of all if

$$b_n = b = \text{const}, \quad \sum_{n=1}^{\infty} c_n = c < \infty, \tag{36}$$

the system (35) becomes identical to (7) with $Z = \sum_{n=1}^{\infty} Z_n$. We now apply the approach from the section 2.2. The equation analogous to (26) here reads

$$\frac{dM_n}{dt} + b_n M_n = c_n \left(\frac{b_n}{2\sigma} - 1 \right) \mid X \mid^2,$$

where

$$M_n = \frac{c_n}{2\sigma} \mid X \mid^2 - Z_n.$$

Hence, at $b_n = 2\sigma$ and $t \to \infty$ we have $M_m \to 0$, so that

$$Z_n(t \to \infty) = \frac{c_n}{2\sigma} \mid X \mid^2$$

Further derivations follow those of section 2.2 with final result equation (31) with the last term on the left hand side multiplied by $c = \sum c_n$. Its contributions depends on the value of c. If one adopts the model speculations from Booty et al. (1982) and assume $c_n = n^{-p}$, $p > 1$, then

$$c = \sum_{n=1}^{\infty} n^{-p} = \zeta(p), \quad p > 1 \tag{37}$$

where $\zeta(p)$ is the Rieman ζ-function. We note that Curry (1978) and Curry et al. (1984) discussed system similar to (35) for the system (1)

4 Another complex Lorenz system

Of the several complex Lorenz-like systems known in the nonlinear geodynamo theories (Weiss et al. 1984, Jones et al. 1985, Roberts and Glatzmaier, 2000), the most interesting appears to be the following

$$\begin{aligned} \frac{dA}{dt} &= \sqrt{\sigma}(\sigma + 1)DB - \sigma A, \\ \frac{dB}{dt} &= iA - \frac{1}{2} iCA^* - B, \\ \frac{dC}{dt} &= -mC - iAB, \end{aligned} \tag{38}$$

where unlike (7) all geodynamo characteristics (phase variables) $A(t)$, $B(t)$, $C(t)$ are complex functions of the kind $K(t) = K_1(t) + iK_2(t)$, $K = A, B, C$, whereas D, σ, m are real physical parameters and in addition $\sigma \propto m \propto 1$. Thus in the real domain (38) is a sixth order nonlinear dynamical system. According to Weiss et al. (1984) it exhibits much richer and more fascinating variety of behaviour than other similar systems discussed in the cited paper. Below we shall concentrate our attention on it.

Following Jones et al. (1985) we set

$$A = \sqrt{2}X, B = (i\sqrt{2}/R)Y, C = 2Z/R, R = i(\sigma+1)D/\sqrt{\sigma}.$$

Then (38) takes the form

$$\begin{aligned} \frac{dX}{dt} &= -\sigma X + \sigma Y, \\ \frac{dY}{dt} &= RX - Y - X^* Z, \\ \frac{dZ}{dt} &= -mZ + XY, \end{aligned} \tag{39}$$

which closely resembles the systems (1) and (7). This fact implies that the subsequent results if not identical, will be similar to those corresponding to (1) and (7). For instance, at $m = 2\sigma$ and $t >> 1/m$ ($t \to \infty$), the equation (31) now reads

$$\frac{d^2X}{dt^2} + (\sigma+1)\frac{dX}{dt} + \sigma(1-R)X + \frac{1}{2} \mid X \mid^2 X = 0. \tag{40}$$

Unlike (31), $\overline{X} = 0$ is the only stationary solution of (40). On the other hand similar to (31) and (34) now

$$X(t) = \delta \exp(i\omega t), \omega = \sqrt{\sigma}D, \delta^2 = 2\delta(D^2 - 1), (D > 1), \tag{41}$$

is an exact periodic solution of (40). In details the stability and the bifurcation properties of the fixed point $\overline{X} = 0$ and of (41) are described in Jones et al. (1985).

The physical constraints $\sigma \propto m \propto 1$ make the previous cases ($\sigma \to \infty$ and /or $m \to 0$) invalid for (39).

5 Summary and conclusion

The main results of this study concern the complex Lorenz-like systems (7) and its infinite Z-component generalization (35) as well as the system (38),

arising in the laser physics and in geophysical fluid dynamics (baroclinic instability theory, geodynamo models, etc.). Some of them recover the results obtained by previous authors. Here we were interested mainly in the asymptotic solutions of the corresponding equations:

- At $\sigma \to \infty$ two decoupled equations (14), (15) of known type are derived from the reduced system (9) - (11) and the solutions (16), (17) are discussed.
- At $t \to \infty$ a single equation (21) with integral (memory) term is derived. In the case of short memory ($1/b \to 0$), equations (22) - (24) are in force.
- At finite σ ($\sigma < \infty$) , the generalized Duffing type equation (31) with exact limit cycle solution (34) is derived. Almost identical to (31) equation (40) is derived for the nonlinear geodynamo model (38). Unlike (31), equation (40) has a zero fixed point only.
- Under the same assumptions, the generalized system (35) is reduced to the standard form (7) with $Z = \sum_{n=1}^{\infty} Z_n$

Our study open possibility for further numerical and analytical investigation of the properties of obtained equations such as investigation of stability properties of attractors, bifurcation diagram and calculation of characteristic quantities such as generalized dimensions . This will be a subject of future research.

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